

Model-order reduction in microresonators using proper orthogonal decomposition

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We study dissipative Kerr-soliton dynamics in optical microresonators using low-dimensional representations of the Lugiato–Lefever equation (LLE). Using proper orthogonal decomposition (POD), we demonstrate that high-dimensional optical states can be accurately represented by a minimal set of energy-ordered modes. We show that this compact basis not only faithfully reconstructs two-soliton interaction dynamics but also provides an interpretable signature for bifurcation analysis, capturing the soliton-to-breather transition without explicit eigenvalue continuation. This approach significantly reduces computational cost while preserving physical insight, offering a transferable protocol for the real-time control and design of complex nonlinear optical systems. © 2026 Optica Publishing Group under the terms of the [Optica Open Access Publishing Agreement](#)

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1. INTRODUCTION

Originating in turbulence analysis and later extensively studied in general-purpose low-dimensional modeling [1], proper orthogonal decomposition (POD) [2] has a long record for building reduced-order models across the physical sciences and engineering [3,4]. POD constructs these models by extracting the most energetic spatial modes from simulation or experimental data. Lately, several POD variants have been developed for control [5], reconstruction [6], inverse and design problems [7], and reduced-basis treatments of parameterized partial differential equations (PDEs) [8]. Within optics, POD applications remain limited; notable examples include POD-based model reductions that capture bifurcation routes and multipulsing transitions in mode-locked lasers to track the onset of oscillatory behavior [9,10].

Here, we apply POD to microresonators, a platform well-suited for tracking optical bifurcations. These devices are distinguished by their ability to confine optical fields within small modal volumes, enhancing light–matter interactions to the point where dispersion, nonlinearity, loss, and gain balance to form stable dissipative Kerr solitons. These self-sustaining soliton wave packets are fundamental to applications ranging from optical frequency comb generation to ultrafast information processing [11–13], making the verification of their operational stability a critical requirement. Consequently, analyzing the complex bifurcation landscapes that govern these stability regimes is indispensable for defining the boundaries of reliable device performance. Bifurcations, from Hopf-induced breathers to period-doubling routes to chaos, not only represent instability but also reveal the underlying mechanisms of nonlinear energy exchange within the cavity [14–16]. Here, we

apply the POD method to the Lugiato–Lefever equation (LLE) [17,18] to obtain a reduced-order representation of soliton dynamics and to use this representation for bifurcation analysis.

Conventional methods for bifurcation analysis typically rely on tracking eigenvalues of large linearized operators or using evolutionary approaches. By contrast, POD extracts dominant structures and their temporal evolution directly from the data and enables the efficient identification of dynamical transitions using a reduced-order approximation. To achieve this reduction, we decompose the spatiotemporal evolution of the intracavity field into a set of spatial and temporal modes using singular value decomposition (SVD). By projecting the full LLE dynamics onto this energy-ordered basis, we observe that the system’s degrees of freedom collapse significantly. A fundamental mode captures the stationary pulse, while a low-rank subspace resolves the repulsive and attractive forces governing two-soliton interaction. Most significantly, the reduced model captures the changes in the phase space associated with the Hopf bifurcation, enabling the detection of breathing dynamics solely through the time evolution of the dominant coefficients. Beyond computational compression, this representation reveals a physically meaningful result: the dynamics of dissipative Kerr solitons, while governed by a high-dimensional PDE, possess low intrinsic dimensionality. The collapse onto a basis of $r = 1–4$ dominant modes indicates that the essential physics of soliton formation, interaction, and bifurcation is governed by a small number of coherent structures. The reduced phase space further provides a geometric picture of dynamical transitions that is more directly interpretable than tracking eigenvalue crossings in a high-dimensional operator spectrum.

To our knowledge, ours is the first systematic use of POD for soliton dynamics in microresonators. The resulting reduced models enable fast reconstructions, efficient parameter sweeps, and compact indicators of regime changes. Finally, because POD relies only on data snapshots, the same methodology applies to other nonlinear optical platforms where real-time monitoring and control benefit from low-dimensional representations. Before closing, we discuss a potential application of our reduced-order model. In particular, drawing an analogy with feedback control in adaptive optics [19–21], the POD-based reduced model can be used to synthesize control signals in a low-dimensional coordinate system, which are then applied to the microresonator through available actuation (e.g., pump or phase modulation). In this setting, measurements from the optical system are mapped onto the dominant POD modes to obtain low-dimensional state estimates, which are used within the reduced model to compute control inputs, thereby closing the feedback loop.

2. BASIC EQUATIONS AND NORMALIZATION

The dynamics of optical fields within microresonators are often described using the LLE, a widely used mathematical model to study nonlinear phenomena in microresonators, particularly the generation and stability of localized structures such as solitons. In general form, the LLE may be written as [22]

$$\frac{\partial \psi}{\partial t} = -(1 + i\alpha)\psi - \frac{iD_2}{2} \frac{\partial^2 \psi}{\partial \theta^2} + i|\psi|^2\psi + F, \quad (1)$$

where $\psi(\theta, t)$ represents the normalized intracavity field envelope, α is the cavity detuning parameter indicating the offset between the driving laser frequency and the nearest cavity resonance, D_2 characterizes the second-order dispersion of the cavity, and F denotes the amplitude of the continuous-wave (CW) external pump field driving the cavity. By letting $x = (1/D_2)^{1/2}\theta$, we can normalize Eq. (1) in terms of D_2 to obtain

$$\frac{\partial \psi}{\partial t} = -(1 + i\alpha)\psi + \frac{i}{2} \frac{\partial^2 \psi}{\partial x^2} + i|\psi|^2\psi + F \quad (2)$$

that describes dynamics in the anomalous dispersion regime. Here, the parameters α and F explicitly appear as free parameters, whereas D_2 is implicitly incorporated in the spatial coordinate. The normalization from $\theta = [-\pi, \pi]$ to $x = [-L/2, L/2]$, where $L = (4\pi^2/|\beta|)^{1/2}$ is the normalized circumference with respect to the dispersive scale length that allows us to explore the limit $L \rightarrow \infty$ [22].

The dynamics in Eq. (2) are highly sensitive to the parameters α and F . As these parameters vary, the system traverses distinct regimes, ranging from stable stationary solitons to oscillatory breathers and chaotic states [22–25]. Here, we apply POD, which is a projection-based model reduction technique that we elaborate on in the next section, to Eq. (2). We structure our analysis in two phases: first, we examine soliton interactions to validate the accuracy of the reduced model in regimes where analytical solutions are available [26,27], and subsequently, we extend the framework to map complex soliton bifurcation landscapes.

3. PROPER ORTHOGONAL DECOMPOSITION

Here, we provide a brief overview of the POD and refer the reader to standard references for further details [10,28,29]. Consider a dynamical system governed by an abstract nonlinear PDE:

$$\mathbf{U}_t = \mathcal{N}(\mathbf{U}, \mathbf{U}_x, \mathbf{U}_{xx}, \dots, x, t), \quad (3)$$

where the state vector $\mathbf{U} \triangleq \mathbf{U}(x, t)$ describes the physical phenomenon of interest, and the subscripts x and t in Eq. (3) denote partial differentiation. The nonlinear operator $\mathcal{N}(\cdot)$ captures the PDE dynamics, and relevant boundary conditions are assumed. Later, we will specialize $\mathcal{N}(\cdot)$ to the operator defining the dynamics in Eqs. (1) and (2). Solutions of Eq. (3) live in an infinite-dimensional space and could be expanded in eigenfunctions of a suitable linearized operator. However, such eigenfunction expansions are impractical and require many modes. Instead, POD enables efficient extraction of the dominant structures directly from the data.

To construct the POD modes, we assemble the simulation data into a matrix $\mathbf{X}$, where each row corresponds to the system state (after spatial discretization) at a particular time. For a dataset containing M time samples, each stored as a vector of length N , we have $\mathbf{X} \in \mathbb{C}^{M \times N}$. The SVD of $\mathbf{X}$ gives $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$, where $\mathbf{U} \in \mathbb{C}^{M \times M}$ and $\mathbf{V} \in \mathbb{C}^{N \times N}$ are unitary matrices, and $\mathbf{\Sigma}$ is the diagonal matrix of singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots \geq 0$. The columns of $\mathbf{V}$, $\{\phi_j(x)\}_{j=1}^N$, are the POD modes and are ordered by the decreasing singular value, which indicate their contribution to the data energy. The relative energy captured by the j th mode is $E_j = \sigma_j^2 / (\sum_{i=1}^N \sigma_i^2)$. Selecting the first r ($\ll N$) modes whose energies account for most of the total yields an efficient truncated basis. Importantly, this truncated basis gives the optimal r -dimensional approximation of the data in a least-squares sense among all orthonormal bases of the same size [28].

The state $\mathbf{U}(x, t)$ is approximated, in a Galerkin sense, as a linear combination of the dominant POD modes: $\mathbf{U}(x, t) \approx \sum_{j=1}^r a_j(t)\phi_j(x)$, where $a_j(t)$ denote the time-dependent complex-valued coefficients. It is convenient to express this linear combination of modes as a matrix–vector product:

$$\mathbf{U}(x, t) \approx \Phi(x)\hat{a}(t), \quad (4)$$

where $\Phi(x) = [\phi_1(x), \phi_2(x), \dots, \phi_r(x)] \in \mathbb{C}^{n \times r}$ is the matrix of POD modes and $\hat{a}(t) = [a_1(t), a_2(t), \dots, a_r(t)]^T \in \mathbb{C}^{r \times 1}$ is the vector of coefficients.

To obtain the reduced-order system representation for the dynamics in Eq. (3), we substitute the POD expansion given by Eq. (4) into Eq. (3) to obtain

$$\frac{d}{dt}[\Phi(x)\hat{a}(t)] = \mathcal{N}(\Phi(x)\hat{a}(t), \Phi_x(x)\hat{a}(t), \Phi_{xx}(x)\hat{a}(t), \dots), \quad (5)$$

where $\Phi_x(x)$ and $\Phi_{xx}(x)$ denote the matrices of the first and second derivatives of the POD modes. We then take the inner product of the resulting expression with each mode ϕ_k , for all $k = 1, \dots, r$, which projects the dynamics onto the reduced basis. This operation yields a system of ordinary differential equations for the modal coefficients. Specifically, for the k th coefficient (with $k \in \{1, \dots, r\}$), we have

$$\frac{da_k}{dt} = \langle \phi_k, \mathcal{N}(\Phi(x)\hat{a}(t), \Phi_x(x)\hat{a}(t), \Phi_{xx}(x)\hat{a}(t), \dots) \rangle, \quad (6)$$

where $\langle \cdot, \cdot \rangle$ is the inner product. We used the orthonormality property ($\langle \phi_j, \phi_k \rangle = \delta_{jk}$) to obtain Eq. (6).

Because the modal matrix $\Phi(x)$ is obtained from the SVD, solving for the modal coefficients in Eq. (6) yields a low-rank approximation of $\mathbf{U}(x, t)$. This completes our short review of the POD method, and we now apply this framework to LLE dynamics in Eqs. (1) and (2).

A. Reduced-Order Spatiotemporal Modes

The core utility of POD lies in its ability to decouple the system's spatial structure from its temporal evolution. Physically, the spatial modes represent the dominant shapes or coherent structures within the cavity. Conversely, the temporal coefficients encode the dynamics, referring to how the energy in that specific shape fluctuates over time. As an example, we show a schematic representation of an experimentally observed breathing soliton [30,31] in Fig. 1(a), and the corresponding dominant spatial and temporal modes in Fig. 1(b). The spatial mode $\phi_1(x)$ captures the fixed, energetic pulse profile, while the temporal mode $a_1(t)$ encodes the time-dependent evolution of that profile. Thus, the breathing behavior is manifested as a periodic fluctuation in pulse amplitude and is captured entirely by the oscillation of the temporal coefficient. This reduction allows the complex spatiotemporal instability to be visualized as a simple limit cycle in the reduced phase space of Fig. 1(c).

4. SOLITON DYNAMICS ANALYSIS USING THE POD METHOD

The POD method begins by approximating the solution $\psi(x, t)$ of Eq. (2) as a superposition of dominant POD modes [29]. For this, we decompose $\psi(x, t)$ in the form of Eq. (4) and plug it back into Eq. (1), which yields

$$\begin{aligned} \frac{d}{dt}[\Phi(x)\hat{a}(t)] = & -(1 + i\alpha)\Phi(x)\hat{a}(t) + \frac{i}{2} \frac{\partial^2}{\partial x^2} [\Phi(x)\hat{a}(t)] \\ & + i|\Phi(x)\hat{a}(t)|^2\Phi(x)\hat{a}(t) + F. \end{aligned} \quad (7)$$

And then following Eq. (6), we may rewrite the LLE in the reduced-order format as

$$\begin{aligned} \left\langle \phi, \phi \frac{d\hat{a}}{dt} \right\rangle = & \left\langle \phi, -(1 + i\alpha)\phi\hat{a} \right\rangle + \left\langle \phi, \frac{i}{2} \frac{\partial^2 \phi}{\partial x^2} \hat{a} \right\rangle \\ & + \left\langle \phi, i|\phi\hat{a}|^2\phi\hat{a} \right\rangle + \langle \phi, F \rangle. \end{aligned} \quad (8)$$

Here, ϕ is of the size $N \times N$ determined by the chosen discretization, where each column represents a spatial mode of the single soliton system, and the inner product in terms of a matrix–vector multiplication operation. Using the orthogonality condition from Eq. (6), Eq. (8) simplifies to

$$\begin{aligned} \frac{d\hat{a}}{dt} = & \left\langle \phi, -(1 + i\alpha)\phi\hat{a} \right\rangle + \left\langle \phi, \frac{i}{2} \frac{\partial^2 \phi}{\partial x^2} \hat{a} \right\rangle \\ & + \left\langle \phi, i|\phi\hat{a}|^2\phi\hat{a} \right\rangle + \langle \phi, F \rangle. \end{aligned} \quad (9)$$

The above equation holds for k -dominant-mode reduction of the LLE based on the system dynamics; meaning, $\hat{a}(t) \in \mathbb{C}^{k \times 1}$, where $k \leq r$. Equation (9) can be solved numerically to obtain $\hat{a}(t)$ for a given step size Δt , where Δt is defined in terms of the photon lifetime of the device.

A. Single-Soliton System ($N_s = 1$)

To numerically obtain a single-soliton solution $\psi_1(x, t)$, we solve the LLE in Eq. (2) in time with $\alpha = 1.9$ and $F = 1.3$. Once we obtain the stationary soliton, we then evolve it for some time t to get the complete evolution profile for modal decomposition to be possible. We consider using the standard split-step Fourier method (SSFM) [32] as our numerical scheme for soliton formation and evolution.

To perform a POD analysis, we apply SVD to the data ensemble matrix $\Psi_1(x, t) = [\psi_1(x, t_0), \psi_1(x, t_0 + \Delta t), \dots, \psi_1(x, t)]$, decomposing it into orthogonal spatial and temporal modes. The resulting singular values are shown in Fig. 2(b), clearly revealing a single dominant singular value. This dominant mode captures virtually all system dynamics, highlighting that a single spatial mode is enough to represent the steady-state soliton evolution accurately. Additional modes correspond to significantly lower-energy singular values, implying negligible contributions, and hence, they can be safely discarded without compromising accuracy. To confirm this, Fig. 2(c) shows the spatial profiles of the first two POD modes obtained via SVD. The first mode closely resembles the shape of the soliton itself, directly associated with the stable stationary

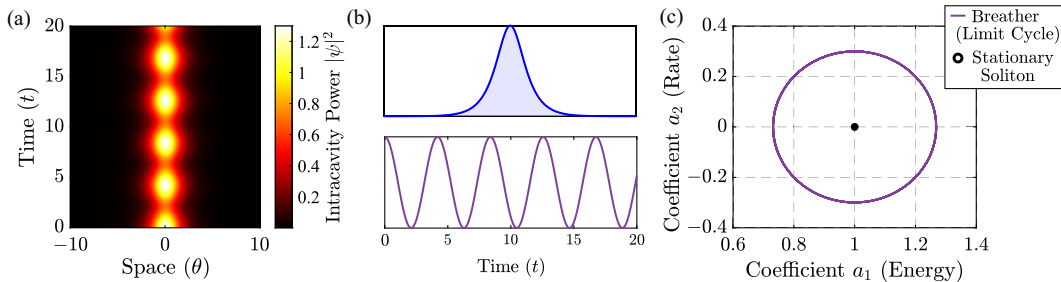


Fig. 1. (a) Schematic of spatiotemporal evolution of a breathing dissipative Kerr soliton as similarly observed in [30,31]. (b) POD separates the dynamics into a dominant energetic spatial mode $\phi_1(x)$ (top) and its corresponding temporal coefficient $a_1(t)$ (bottom). (c) Projection onto the reduced phase space (a_1 versus a_2). The stationary soliton appears as a stable fixed point (black dot), while the breathing dynamics trace a limit cycle (red loop).

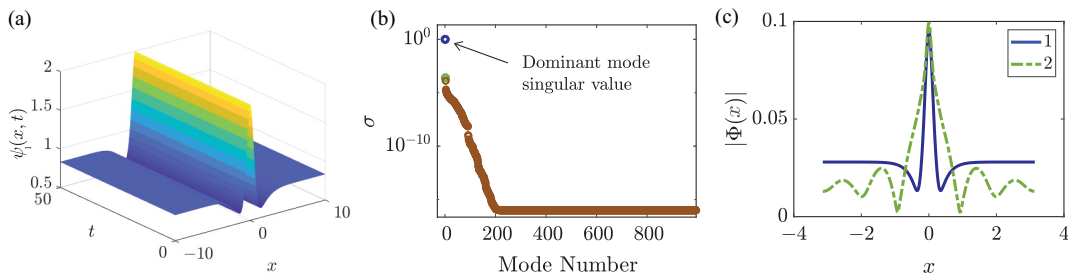


Fig. 2. (a) Evolution of a single-soliton solution. (b) Singular values obtained through the SVD analysis of (a), showing the most dominant singular value in blue and the second most dominant one in green. (c) Spatial profile corresponding to the two dominant singular values, where the first one resembles the shape of the soliton.

structure, whereas the second mode, along with subsequent modes, represents residual spatial structures irrelevant to the steady-state soliton behavior. This shows that the POD analysis explicitly justifies the use of a reduced-order, single-mode model for representing the dynamics of a stationary single soliton.

B. Two-Soliton System ($N_s = 2$)

We then apply the POD method to systems that require multiple modes to accurately capture the dynamics, such as a two-soliton molecule system. In the large detuning limit, for chosen parameters $\alpha = 1.9$ and $F = 1.3$, we derived an analytical expression for the potential well function formed between two solitons in our previous work [26,27]. Here, we consider modeling the same interaction dynamics using the POD technique that enables verification of this reduced-order modeling. We consider the initial condition to be

$$\psi_2(x, t) = \psi_1(x - x_{ph}, t) + \psi_1(x + x_{ph}, t) + \psi_c(x, t) + \phi(x, t). \quad (10)$$

Here, x_{ph} is half the separation distance between solitons; $\psi_1(x, t)$ is the single-soliton solution of the LLE obtained by evolving the system using the SSFM with periodic boundary

conditions; $\psi_c(x, t)$ is the common pedestal shared between the two solitons; and $\phi(x, t)$ is the localized adjustment term required to form a two-pulse solution.

From our previous work, we particularly analyze a scenario at a separation $x_{ph} = 1.4$ where the two solitons collapse and merge into a single-soliton solution due to strong attractive interaction [Fig. 3(a)], illustrating a system dominated by strong nonlinear effects. This separation lies outside the perturbative regime of the analytical interaction theory [26,27], where the soliton pedestals are assumed to be weakly overlapping. We choose this strongly interacting case to test the POD method under conditions where the analytical description no longer applies. To accurately reconstruct the collision dynamics via the POD technique, the model requires $r_0 + r_n$ modes. We note that although strong nonlinear transients are present during the collapse, the chosen parameters place the system within the soliton interaction regime; no Turing-roll or chaotic dynamics are involved in this process. Here, r_0 represents the dominant fundamental mode, while r_n denotes the higher-order modes essential for maintaining dynamical fidelity. We first apply SVD to $\Psi_2(x, t) = [\psi_2(x, t_0), \psi_2(x, t_0 + \Delta t), \dots, \psi_2(x, t)]$, which is constructed by storing the time-evolution data of

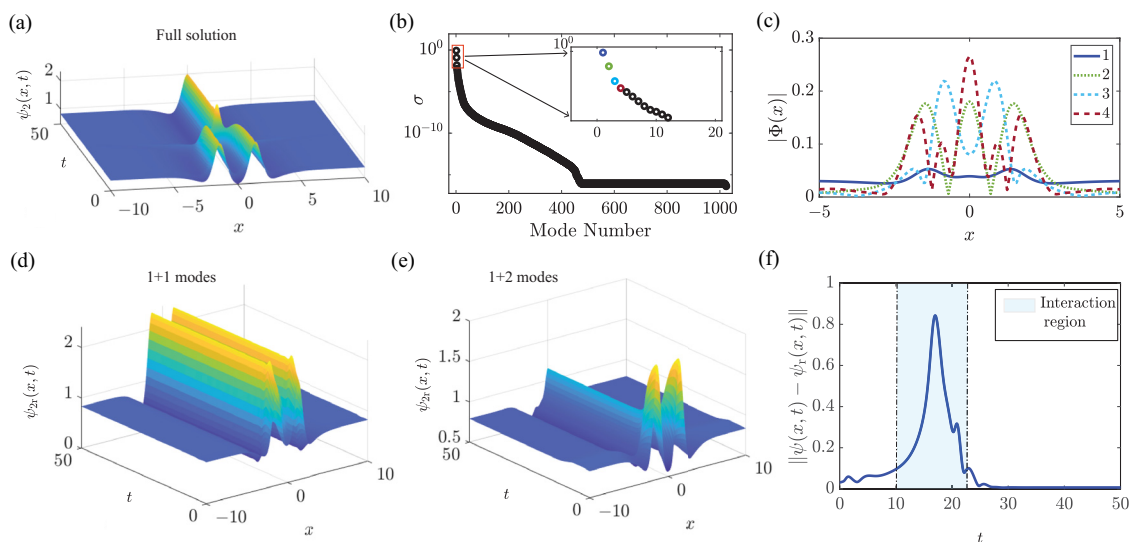


Fig. 3. (a) Evolution in the temporal domain showing the two-soliton collapse and formation of a single soliton. (b) Singular values obtained through the SVD analysis of (a), where the most dominant singular values are shown in the inset. (c) Spatial profile corresponding to the four most dominant singular values. (d), (e) Modeling the interaction phenomena shown in (a) with two and three modes. (f) Difference in norm between the full soliton $\psi_2(x, t)$ and the reduced solution $\psi_{2r}(x, t)$ obtained by the POD technique with $r_0 + r_n (= 1 + 3)$ modes in the temporal domain.

the two-soliton collision. We display the resulting singular values in Fig. 3(b), highlighting the dominant singular values in the inset. Analyzing the spatial modes reveals that there are $r_0 + r_n (= 1 + 3)$ dominant modes significantly contributing to the temporal evolution of $\psi_2(x, t)$, as shown in Fig. 3(c). Each of these modes is essential for accurately reconstructing the transition from a two-soliton configuration into a single-soliton system. The necessity of including $1 + 3$ modes is justified by Figs. 3(d) and 3(e), which illustrate how retaining only $1 + 1$ or $1 + 2$ modes leads to an inaccurate reconstruction of $\psi_2(x, t)$. As we proceed with $1 + 3$ modes, we reconstruct the system behavior accurately [not shown here for avoiding repetition of Fig. 3(a)] and show the difference between the full solution $\psi(x, t)$ and the reconstructed solution $\psi_{2r}(x, t)$ in Fig. 3(f) in the temporal domain.

The difference is significant in regions where collapse occurs due to strong nonlinear interactions between solitons, and it subsequently decreases as a single soliton is formed. This discrepancy arises because POD relies on the linear superposition of modes, as described in Eq. (4), while the strong nonlinearity present in the system introduces deviations between the full and reduced solutions. However, this is not a limitation, as the method still successfully captures the collapse and formation of the single soliton as the two solitons interact.

We also examined another case from our analytical work [26] where the interaction between solitons is repulsive, corresponding to $\alpha = 2.7$ and $F = 1.98$. Here, the nonlinear interaction decreases as x_{ph} increases, allowing the system to be accurately represented by a linear combination of reduced-order modes. Figure 4(a) illustrates the repulsive interaction between the solitons, and applying POD to the full solution yields the singular values and spatial modes presented in Fig. 4(b). We then reconstruct $\psi(x, t)$ using the $1 + 3$ dominant modes and present the convergence error between the full and reconstructed solutions in Fig. 4(c), where the error is significantly lower than in the previous case. By comparing these two types of interaction dynamics, we conclude that the POD method achieves better convergence and requires fewer modes in cases with weaker nonlinearity.

The qualitative POD structure, including the dominant mode count and spatial character, is robust to small variations in α and F within a given dynamical regime. However, as these parameters are varied significantly, the system may enter a different dynamical regime, and a new POD decomposition is required to capture the corresponding coherent structures.

The reduced-order basis is therefore regime-specific rather than globally universal across the full LLE parameter space.

5. POD AND SOLITON BIFURCATION

Microresonator dynamics governed by the LLE exhibit different types of bifurcations, including saddle-node, Hopf, and period-doubling transitions, which signal the emergence of steady states or the onset of oscillatory behavior [14,25,33]. Among these, Hopf bifurcations are especially relevant here, as they govern the transition from stationary to breathing solitons [34]. Our aim is to determine whether the low-dimensional dynamics obtained from the POD method accurately capture this bifurcation behavior.

We begin by computing a stationary single-soliton solution ($\psi_0(x)$) using the SSFM evolution for $\alpha = 3.479$, $\beta = -0.1549$, and $F = 2.56$. Figure 5(a) shows the eigenvalue spectrum of the Jacobian matrix obtained by linearizing the LLE at $\psi_0(x)$. This spectrum exhibits fourfold symmetry, with a zero eigenvalue arising from the system's translational invariance. All eigenvalues with negative real parts indicate the stability of the stationary solution $\psi_0(x)$. In a Hopf bifurcation, a conjugate pair of complex eigenvalues crosses the imaginary axis from the left-half plane (stable) to the right-half plane (unstable) as F is varied in Eq. (1). This crossing of the real part through zero indicates the bifurcation point.

As the parameter F is increased beyond the value that yields $\psi_0(x)$, the system approaches and begins to cross the stability boundary. This transition is marked by a shift in the real part of the eigenvalues toward positive values, indicating the onset of instability. At $F = 2.7$, the system undergoes a Hopf bifurcation [23]. This is indicated by the real parts of two previously close-to-zero eigenvalues becoming positive, leading to the formation of a breather solution as shown in Fig. 5(b). By tracking the solution $\psi(x, t)$ as it evolves through the bifurcation, we gather the necessary time-series data. Subsequently, we perform POD on this dataset, employing SVD to decompose the solution into its fundamental spatial ($\mathbf{U}$) and temporal ($\mathbf{V}^*$) modes. In this instance, the investigation of the soliton bifurcation requires a reduced-order model based on only $1 + 2$ dominant modes for accurate reconstruction. The effectiveness of the POD technique becomes evident when observing the three dominant temporal modes; see Fig. 5(c). Their evolution provides a clear and direct visualization of the transition from a soliton to a breather, which would not be possible otherwise.

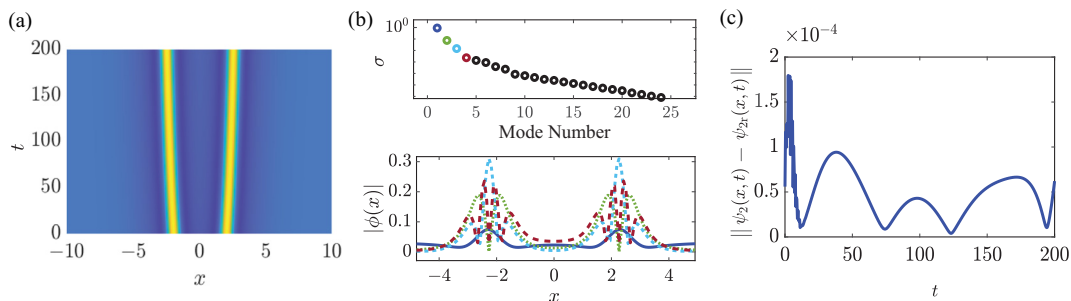


Fig. 4. (a) Repulsive interaction in the two-soliton system. (b) Singular values and the corresponding spatial modes for the four dominant modes. (c) Difference between the full solution and the reduced-order solution in the temporal domain.

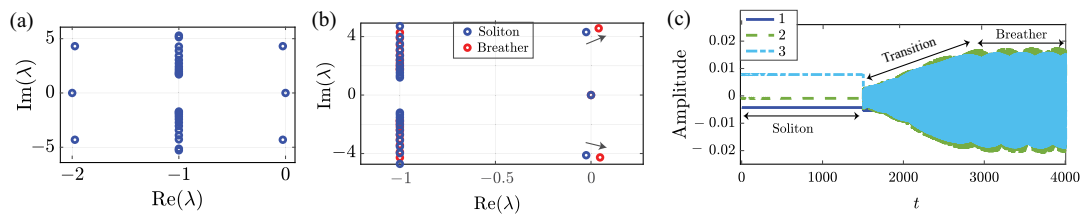


Fig. 5. (a) Eigenvalue profile of a single-soliton system obtained by linearization of the LLE. (b) Observation of the Hopf bifurcation in the eigen-spectrum as the soliton-to-breather transition takes place. (c) Temporal modes obtained by the SVD operation.

The POD offers a reduced-order representation of the complex dynamics involved in the bifurcation and also allows for a clearer understanding of how the localized soliton state evolves into the oscillatory breather state, pinpointing the critical modes responsible for the Hopf bifurcation observed.

6. NUMERICAL CHALLENGES

While model-order reduction has been successfully applied to nonlinear optical propagation, existing studies have largely focused on the nonlinear Schrödinger (NLS) equation and its variants in conservative fiber systems [10,28]. Adapting this framework to microresonator dynamics governed by the LLE presents distinct challenges due to the fundamentally different mathematical structure of the problem.

Unlike the NLS, the LLE is a stiff, driven-dissipative partial differential equation. The stiffness arises from the presence of the dissipative loss term in the equation. A standard POD approach applied to smooth, energy-conserving NLS breathers does not automatically guarantee convergence in the dissipative LLE regime. While modeling soliton dynamics via POD, the selection of the temporal sampling interval is critical; a sufficiently fine step size (Δt) is required to resolve the rapid transient features essential for accurate mode reconstruction. Here, we implemented the SSFM with an adaptive time-stepping scheme. This approach ensures the resolution of critical transient dynamics while minimizing computational overhead. Prior implementations of POD for soliton dynamics have relied on fixed-step Runge–Kutta integration [10]. A critical insight emerging from our analysis is the necessity for numerical consistency: the scheme used to generate the snapshot ensemble must be identical to the one used for evolving the reduced temporal coefficients $a(t)$. We observe that mixing distinct integration methods introduces numerical artifacts that degrade the possibility of the temporal mode reconstruction.

A. Computational Advantages

The computational advantage of the POD framework is best understood through the standard offline–online decomposition. In the offline phase, the full SSFM solution is computed once to generate the snapshot ensemble, and SVD is applied to extract the dominant spatial modes. This step carries the same cost as a conventional simulation. Once the basis is established, all subsequent analyses are carried out by evolving a reduced system of r ordinary differential equations [Eq. (9)] rather than the full $N = 1024$ dimensional discretized PDE. In the cases studied here, $r = 4$ (two-soliton interaction) and $r = 3$ (breather

transition) yield a reduction in the system's degrees of freedom by a factor of approximately 250–340. The soliton-to-breather transition manifests directly as a qualitative change in the temporal POD coefficients. This change takes the form of a shift from a fixed point to a limit cycle, and it provides a lightweight indicator of bifurcation. Explicit eigenvalue tracking of the full Jacobian is not required for this detection. We note that eigenvalue analysis remains valuable for validation [Fig. 5(b)]. However, the temporal POD coefficients alone are sufficient to detect and monitor the onset of the transition in practice.

7. CONCLUSION

We have demonstrated that POD serves as a robust framework for compressing the high-dimensional dynamics of microresonator frequency combs into compact, interpretable models. Our results establish that the complexity of the LLE can be drastically reduced without sacrificing physical accuracy. In summary, a single energetic mode suffices to reconstruct stationary solitons, while a sparse low-rank basis captures the nonlinear scattering forces of multi-soliton interactions. Most significantly, we show that the transition to the oscillatory breathing state, which traditionally requires computationally expensive eigenvalue tracking, manifests naturally as a limit cycle in the reduced temporal coefficients.

By decoupling the spatial structure from temporal evolution, this approach offers a lightweight alternative for stability analysis that is fast enough for real-time applications. Looking forward, this framework lays the groundwork for bifurcation analysis in nonlinear systems, enabling predictive control and rapid parameter space mapping not only in microresonators but also in fiber Kerr cavities, mode-locked lasers, and optomechanical systems. Extending the POD framework to Turing-roll and chaotic regimes of the LLE, where the dominant coherent structures are less localized, represents an interesting direction for future investigation.

Disclosures. The authors declare no conflicts of interest.

Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

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