Statistical Analysis of the Performance of PMD Compensators Using Multiple Importance Sampling

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***Abstract*—In this letter, we evaluate the performance of fixed and variable differential group delay (DGD) polarization-mode dispersion (PMD) compensators as the first- and second-order PMD varies using multiple importance sampling. We show that importance sampling yields estimates of the average penalty with low variance over the entire region of interest of first- and second-order PMD. We also show that there is little advantage in using a compensator with a variable-DGD element and that the performance of a compensator that minimizes the residual DGD at the central frequency of the channel is considerably worse than a compensator that maximizes the eye opening.**

***Index Terms*—Monte Carlo methods, optical communications, outage probability, polarization-mode dispersion (PMD).**

I. Introduction

POLARIZATION-MODE dispersion (PMD) is one of the barriers to upgrading the current per-channel data rates to 10 Gb/s and beyond in a large number of terrestrial optical fiber transmission systems. Therefore, in recent years a considerable effort has been devoted to mitigating the effects of PMD, based on optical, electrical, and optoelectrical compensators [1]–[3]. Many performance studies of compensators have focused on the average pulse spreading reduction, and hence, the average bit-error rate (BER) of optical systems. However, reducing the average BER may not significantly reduce the outage probability in the range of $10^{-6} - 10^{-5}$ where real systems must operate [2]. Previous studies relying on average BER reduction have also not addressed how the penalty is explicitly related to the first- and the second-order PMD.

In this contribution, we evaluate the performance of single-section PMD compensators in a large region of the plane of the magnitude of first- and second-order PMD using multiple importance sampling applied to first- and second-order PMD [4]. As our performance measure, we use the average value of the eye-opening penalty as a function of the magnitude of the first- and second-order PMD.

The use of multiple importance sampling applied to PMD allows one to efficiently study important rare events with large first- and second-order PMD. Therefore, one can accurately calculate outage probabilities for the PMD-induced penalty on the order of 10^{-5} or less in compensated or uncompensated systems. We note that third- and higher order PMD are also included in our simulations, but we do not specifically bias our simulations toward values of third- and higher order PMD other than the moderately large values that appear naturally when one biases the first- and second-order PMD values. In this study, we extend the work in [3], where only outage probabilities larger than 10^{-4} could be efficiently computed due to the use of standard unbiased Monte Carlo simulations. We also extend the work in [2], where importance sampling was only applied to first-order PMD.

In order to compensate for PMD, we consider two types of single-section PMD compensators. The first type consists of a polarization controller (PC) followed by a polarization-maintaining fiber that has a fixed differential group delay (DGD) element. The second type also uses a PC, but has a variable-DGD element. The parameters of the PC's orientation are the only free parameters that a compensator with a fixed-DGD element possesses, while the value of the DGD is an extra free parameter that the variable-DGD compensator has to control. The simplicity of the implementation of these compensators and their reduced number of free parameters to control, as opposed to compensators with multiple sections, make them attractive as PMD compensators.

II. Theory

In this letter, we study a 10-Gb/s nonreturn-to-zero system with a mean DGD of 30 ps. The fiber model uses 80 sections of birefringent fiber with the coarse step method, which reproduces first- and higher order PMD distortions within the probability range of interest. Our results may be applied to 40-Gb/s systems by scaling down the time quantities by a factor of four.

For performance evaluation, we use the eye opening, which is defined as the difference between the currents in the lowest mark and the highest space at the decision time. To define the decision time, we recovered the clock using an algorithm based on one described in [5]. The eye-opening penalty is the ratio between the back-to-back and the PMD-distorted eye opening. We compute the joint probability density function (pdf) of the magnitude of first- and second-order PMD, $|\boldsymbol{\tau}|$ and $|\boldsymbol{\tau}_\omega|$, using multiple importance sampling applied to first- and second-order PMD [4], and the average value of eye-opening penalty given a value of $|\boldsymbol{\tau}|$ and $|\boldsymbol{\tau}_\omega|$, where the subscript ω represents the derivative with respect to the angular frequency ω.

The main idea of Monte Carlo simulations with multiple importance sampling is that multiple biased simulations are used to generate arbitrary combinations of first- and second-order PMD, effectively covering the regions of the $|\boldsymbol{\tau}| - |\boldsymbol{\tau}_\omega|$ plane of statistical significance. The approach that we use closely resembles that of [4]. To adequately cover the $|\boldsymbol{\tau}| - |\boldsymbol{\tau}_\omega|$ plane, we combine nine biased simulations and one unbiased simulation with 10^5 samples each, using balanced heuristics [4]. For

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each fiber realization, we compute $|\boldsymbol{\tau}|$, $|\boldsymbol{\tau}_\omega|$, and the eye-opening penalty. Dividing the $|\boldsymbol{\tau}| - |\boldsymbol{\tau}_\omega|$ plane into $25 \times 25 = 625$ evenly spaced bins, we then determine the estimator of the average eye-opening penalty value and the expected variance of this estimator in each bin, using the estimators

$$\hat{\mu} = \frac{\hat{M}}{\hat{C}}, \quad \hat{\sigma}_{\hat{\mu}}^2 = \frac{\hat{\sigma}_{\hat{M}}^2}{\hat{C}^2} + \frac{\hat{\mu}^2 \hat{\sigma}_{\hat{C}}^2}{\hat{C}^2} \tag{1}$$

where $\hat{C} = \sum_{j=1}^{J} \hat{C}_j$, $\hat{M} = \sum_{j=1}^{J} \hat{M}_j$ with $\hat{C}_j = \frac{1}{N_j} \sum_{i=1}^{N_j} I(\mathbf{x}_{ij}) w_j(\mathbf{x}_{ij}) L_j(\mathbf{x}_{ij})$ $\hat{M}_j = \frac{1}{N_j} \sum_{i-1}^{N_j} f(\mathbf{x}_{ij}) I(\mathbf{x}_{ij}) w_j(\mathbf{x}_{ij}) L_j(\mathbf{x}_{ij})$. Using (1), we can generate one-standard-deviation confidence intervals for the average penalty in each bin, which is computed using importance sampling. N_j is the number of samples drawn from the jth biased distribution $p_j^*(\mathbf{x})$. The vector $\mathbf{x}_{ij}$ is the set of biased parameters in the ith fiber realization of the jth distribution, where $f(\mathbf{x}_{ij})$ is the associated eye-opening penalty, and J is equal to ten, which comprises nine biased and one unbiased simulation. The likelihood ratio of the ith fiber realization in the jth biased distribution is $L_j(\mathbf{x}_{ij}) = p_j(\mathbf{x}_{ij})/p_j^*(\mathbf{x}_{ij})$, where $I(\mathbf{x}_{ij})$ is the indicator function whose value is one in a bin of interest and zero outside this bin, and $w_j(\mathbf{x}_{ij})$ are the weights associated with each individual distribution [4]. The distributions $p_j(\mathbf{x})$ and $p_j^*(\mathbf{x})$ are the unbiased and biased pdfs of a random vector $\mathbf{x}$, respectively. Finally

$$\hat{\sigma}_{\hat{C}}^2 = \sum_{j=1}^{J} \frac{1}{N_j(N_j-1)} \sum_{i=1}^{N_j} \left[I(\mathbf{x}_{ij}) w_j(\mathbf{x}_{ij}) L_j(\mathbf{x}_{ij}) - \hat{C}_j \right]^2 \tag{2}$$

$$\hat{\sigma}_{\hat{M}}^2 = \sum_{j=1}^{J} \frac{1}{N_j(N_j-1)} \times \sum_{i=1}^{N_j} \left[f(\mathbf{x}_{ij}) I(\mathbf{x}_{ij}) w_j(\mathbf{x}_{ij}) L_j(\mathbf{x}_{ij}) - \hat{M}_j \right]^2. \tag{3}$$

It is important to note that the estimators for the mean eye-opening penalty $\hat{\mu}$ and the variance $\hat{\sigma}_{\hat{\mu}}^2$ of the estimator $\hat{\mu}$ given by (1) are biased estimators. We obtain the expression for the variance by using the law of propagation of errors, where in first-order approximation $\hat{\sigma}_{\hat{\mu}}^2/\hat{\mu}^2 = \hat{\sigma}_{\hat{M}}^2/\hat{M}^2 + \hat{\sigma}_{\hat{C}}^2/\hat{C}^2$. The bias in the estimation of $\hat{\mu}$ and $\hat{\sigma}_{\hat{\mu}}^2$ can be reduced by computing $\hat{C}$ with a much larger number of samples than are used to compute $\hat{M}$. This approach is practical because the computational cost of generating fiber realizations to calculate $\hat{C}$ is much smaller than the cost to compute penalties after compensation, which are required to compute $\hat{M}$. Here, we reduce the uncertainty in $\hat{C}$ by computing $\hat{C}$ using 10^7 samples per biased simulation, while $\hat{M}$ is computed using 10^5 samples per biased simulation. Using this approach, we note that $\hat{\sigma}_{\hat{C}}^2/\hat{C}^2$ is, in general, two orders of magnitude smaller than $\hat{\sigma}_{\hat{M}}^2/\hat{M}^2$. The maximum value for $\hat{\sigma}_{\hat{C}}^2/\hat{C}^2$ is 1.21×10^{-4}. The maximum value for $\hat{\sigma}_{\hat{M}}^2/\hat{M}^2$ is 0.14, but it is much smaller in almost all bins, typically around 0.001 and 0.002.

III. Results

Fig. 1 shows contour plots (dotted lines) of the joint pdf of the magnitude of first- and second-order PMD, $|\boldsymbol{\tau}|$ and $|\boldsymbol{\tau}_\omega|$ for an uncompensated system, which have been obtained as in [4].

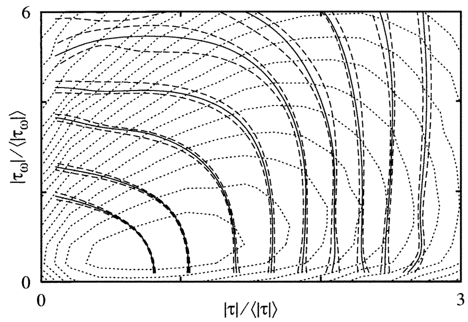

Fig. 1. Uncompensated system. The dotted lines are the contour plots of the joint pdf of the normalized first- $|\boldsymbol{\tau}|$ and second-order PMD $|\boldsymbol{\tau}_\omega|$. The solid and the dashed lines are the contour plots of the conditional expectation of the eye-opening penalty and the confidence interval of the contour plots, respectively. The contours of the joint pdf are at 3×10^{-N}, $N = 1, \ldots, 7$ and 10^{-N}, $N = 1, \ldots, 11$. The curves of the conditional expectation of the eye-opening penalty in decibels are at 0.1, 0.2, 0.4, 0.6, 0.9, 1.2, 1.6, 2.2, and 3.2.

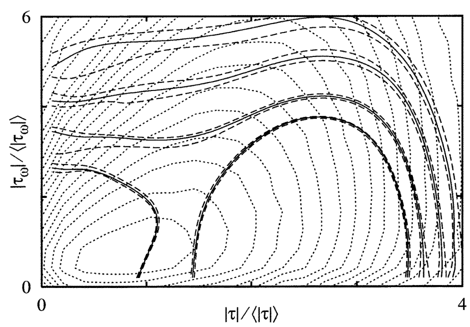

Fig. 2. Same set of curves in Fig. 1 for a compensated system with a fixed-DGD compensator with constant DGD element equal to 2.5 $\langle|\boldsymbol{\tau}|\rangle$. The penalty curves in decibels are at 0.1, 0.2, 0.3, and 0.4.

We also show contours of the eye-opening penalty (solid lines) for an uncompensated system and the eye-opening penalty with one-standard-deviation added and subtracted (dashed lines). These three sets of curves are then smoothed using an Nth-order Bezier approximation, where N is the number of points in the contour. The dashed lines represent the one-standard-deviation confidence intervals for the penalty, given by $\hat{\sigma}_{\hat{\mu}}$ in (1), and are quite narrow except at the edges of the plot, demonstrating the effectiveness of importance sampling in reducing the variance of the estimator of the penalty in this case. It is important to note that the region of the $|\boldsymbol{\tau}| - |\boldsymbol{\tau}_\omega|$ plane that is the dominant source of a given penalty is where the corresponding penalty level curve intersects the contour of the joint pdf of $|\boldsymbol{\tau}|$ and $|\boldsymbol{\tau}_\omega|$ with the highest probability. The contour plots for penalties beyond 1.2 dB are approximately parallel to the second-order PMD axis, indicating the expected result that first-order PMD is the dominant cause of penalty in this uncompensated system.

Figs. 2–4 show the contours of the eye-opening penalties when different PMD compensators are used. The eye-opening penalty contours are plotted as a function of the uncompensated $|\boldsymbol{\tau}|$ and $|\boldsymbol{\tau}_\omega|$, and we show the same contours of their joint pdf, as in Fig. 1. Fig. 2 shows the eye-opening penalty with a fixed-DGD compensator with a 75-ps DGD element, in which the polarization transformation produced by the PC has been optimized to maximize the eye opening. Fig. 3 shows the eye-opening penalty with a variable-DGD compensator, in

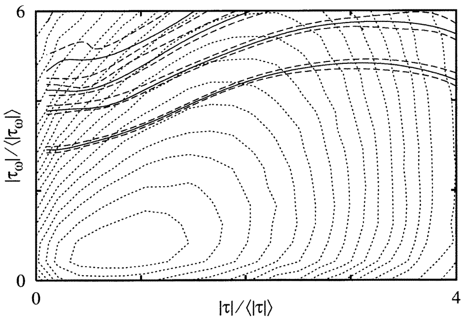

Fig. 3. Same set of curves in Fig. 1 for a compensated system with a variable-DGD compensator with eye opening maximization. The penalty curves in decibels are at 0.1, 0.2, 0.3, and 0.4.

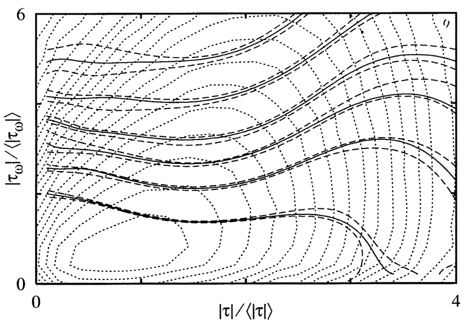

Fig. 4. Same set of curves in Fig. 1 for a compensated system with a variable-DGD compensator with minimized DGD after compensation at the central frequency of the channel. The solid lines are the contours of the penalty in decibels, at 0.1, 0.2, 0.3, 0.4, 0.6, and 0.9.

which, once again, the eye opening has been maximized. Fig. 4 shows the eye-opening penalty with a variable-DGD compensator, in which the residual DGD of the system at the central frequency of the channel has been minimized after compensation.

The first observation that we make, comparing Figs. 2 and 3, is that for penalties above 0.2 dB, the performance of the fixed-DGD compensator is comparable to the variable-DGD compensator as long as $|\boldsymbol{\tau}|/\langle|\boldsymbol{\tau}|\rangle \leq 3.0$. The domain $|\boldsymbol{\tau}|/\langle|\boldsymbol{\tau}|\rangle > 3.0$ corresponds to an outage probability in the uncompensated system of less than 10^{-7}, which is usually negligible. As expected, we also observe that the penalty with the variable-DGD compensator is dominated by higher order PMD. We infer this result by noting that the contour lines of the penalty are nearly parallel to the $|\boldsymbol{\tau}|/\langle|\boldsymbol{\tau}|\rangle$ axis, indicating that the penalty is nearly independent of $|\boldsymbol{\tau}|$. Perhaps a bit more surprisingly, we observe the same result with the fixed-DGD compensator as long as the penalty is above 0.2 dB and $|\boldsymbol{\tau}|/\langle|\boldsymbol{\tau}|\rangle \leq 3.0$. Comparing Figs. 3 and 4, we observe that a variable-DGD compensator that minimizes the residual DGD performs significantly worse than a compensator that maximizes the eye opening. This result indicates once again the importance of higher order PMD in the compensator performance.

In Fig. 5, we plot the outage probability $(\hat{P}_o)$ as a function of the eye-opening penalty for the compensators that we study. The outage probability is the complement of the cumulative density function (cdfc) of the eye-opening penalty ρ, where cdfc $(\rho) = \int_\rho^\infty p(\rho')d\rho'$, and $p(\rho)$ is the corresponding pdf. The maximum relative error $(\hat{\sigma}_{\hat{P}_o}/\hat{P}_o)$ for the curves shown

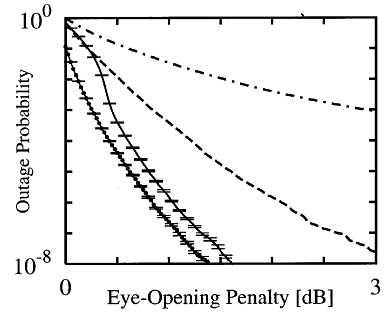

Fig. 5. Outage probability as a function of the eye-opening penalty margin. The outage probability $(\hat{P}_o)$ is the probability that the penalty exceeds the value displayed on the horizontal axis. 1) Dashed–dotted line: uncompensated case. 2) Dashed line: variable-DGD compensator with the compensated DGD minimized at the central frequency of the channel. 3) Solid line: fixed-DGD compensator with DGD element equal to 2.5 $\langle|\boldsymbol{\tau}|\rangle$ and maximized eye opening. 4) Solid–dotted line: variable-DGD compensator with maximized eye opening. The error bars show the confidence interval for the curves that have at least one bin whose relative error $(\hat{\sigma}_{\hat{P}_o}/\hat{P}_o)$ exceeds 10%. For those curves, we show the error bars for one out of three consecutive bins.

in this plot equals 0.14. This plot confirms the results that we inferred from Figs. 2–4. The performance of the fixed- and variable-DGD compensators is comparable. The performance of a variable-DGD compensator that minimizes the residual DGD is significantly worse than the performance of a variable-DGD compensator that maximizes the eye opening. This result, which demonstrates the importance of higher order PMD in determining the penalty, is consistent with [6], where it is shown that a feedback signal provided by a frequency-selective polarimeter is better correlated to the PMD-induced penalty when extracting more values of the polarization dispersion vector over the spectrum of the signal. Because second- and higher order PMD dominate the penalties after compensation, biasing the first-order PMD alone, as was done in [2], does not yield correct quantitative values for the penalty because the statistical variances are large, although the qualitative results remain correct. This issue will be discussed in detail in a future publication. Here, we simply note the importance of calculating the expected statistical variance, as we do throughout this letter.

The outage probability curve referring to a variable-DGD compensator does not equal 1 at 0 dB because higher order PMD in the transmission line will chirp the pulses. In most cases, the DGD of the line is small and the variable-DGD element of the compensator can then compress the pulses, which produces an eye opening at the sampling time that is larger than in the back-to-back case.

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