III. Propagation

A. Basic Wave Equation

1. The basic equation that governs <u>linear</u> propagation of the wave envelope may be written

$$i\,\frac{\partial\, \tilde{u}(z,\omega)}{\partial z} + \left[\beta(\omega_0+\omega) - \beta_0 - \beta_0'\,\omega\right]\tilde{u}(z,\omega)$$

where

Note: I am defining here conventions that are the same as in OCS. These conventions are consistent with Agrawal who uses the physics convention with β not k for the wavenumber

$\underline{\omega}$ is the frequency offset with respect to ω_0 (our central frequency)

$\underline{\beta_0}$ is $\beta(\omega_0)$, $\underline{\beta_0'}$ is $\partial\beta/\partial\omega|_{\omega_0}$

$$\underline{\tilde{u}(z,\omega)} \equiv \frac{1}{T}\int_0^T u(z,t)\exp(i\omega t)\,dt$$

This definition implies a finite time window of size T

In this case:

$$u(z,t) \equiv \frac{T}{2\pi}\int_0^{\Omega} \tilde{u}(z,\omega)\exp(-i\omega t)\,d\omega$$

These are only Fourier transform pairs in the limit $T \to \infty$. This definition is natural. If we don't divide by T, $\tilde{u} \to \infty$.

where $\Omega = 2\pi N/T$ and

N is the number of points in the discretization.

The advantage of this definition (as opposed to the "standard" definition without T) is that both $|\tilde{u}(z,\omega)|^2$ and $|u(z,t)|^2$ have units of power.

6. <u>IMPORTANT NOTE:</u>

As soon as we speak of of a finite time and a finite frequency window $u(z,t)$ and $\tilde{u}(z,\omega)$ MUST be discretized.

So, really: $$\tilde{u}(z,\omega_n) = \frac{1}{T}\sum_{m=0}^{N-1} \underset{\text{///}}{dt} \cdot u(z,t_m)\exp(i\omega_n t_m)\cdot\frac{2\pi}{\Omega}$$

or $$\tilde{u}(z,\omega_n) = \frac{1}{N}\sum_{m=0}^{N-1} u(z,t_m)\exp(i\omega_n t_m)$$

$$u(z,t_m) = \sum_{m=0}^{N-1} \tilde{u}(z,\omega_n)\exp(-i\omega_n t_m)$$

[since $d\omega = 2\pi/T$]

We also have

$$\text{Total Energy} = T \sum_{m=0}^{N-1} |\tilde{u}_m|^2 = \sum_{m=0}^{N-1} |u_m|^2 \frac{2\pi}{\Omega}$$

$$\text{Average Power} = \sum_{m=0}^{N-1} |\tilde{u}_m|^2 = \frac{1}{N} \sum_{m=0}^{N-1} |u_m|^2$$

$$\text{Power Spectral Density} = \frac{|\tilde{u}_m|^2}{\Delta\nu} = \frac{2\pi |\tilde{u}_m|^2}{\Delta\omega}$$

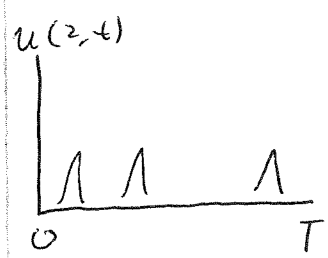
$$= |\tilde{u}_m|^2 \cdot T$$

c. We note also

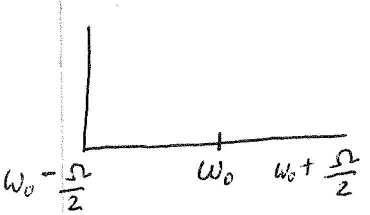
$$t = \tau - \beta_0' z$$

where τ is <u>physical time</u>
so t is <u>retarded time</u>

d. In OCS, where we deal with streams of pulses, the time origin is at the left of the "box".



However, ω_0 is in the center of the "box"



Hence: $\omega_{\frac{N}{2}} = 0$

$$\omega_m = \begin{cases} \Delta\omega (m - N) & \frac{N}{2} \le m \le N-1 \\ \Delta\omega \cdot m & 0 \le m \le N/2 - 1 \end{cases}$$

This point is only important when plotting because

$$\exp(i\omega_m t_n) = \exp[i(\omega_m + N\Delta\omega)t_n]$$

because $N\Delta\omega t_n = N \cdot \frac{2\pi}{T} \cdot \frac{(n-1)}{N} \cdot T$

$$= 2\pi(n-1)$$

2. Defining $\Delta\beta(\omega) = \beta(\omega_0+\omega) - \beta_0 - \beta_0'\omega$ our basic wave equation becomes

$$i\frac{\partial \tilde{u}(z,\omega)}{\partial z} + \Delta\beta(\omega)\,\tilde{u}(z,\omega) = 0$$

a. Generally, to transform into the time domain, we have a convolution

$$i\frac{\partial u(z,t)}{\partial z} + \frac{1}{T}\int_0^T I[\Delta\beta](\tau)\, u(z,t-\tau)\,d\tau = 0$$

where $I[\Delta\beta](t) \equiv \frac{T}{2\pi}\int_0^\Omega \Delta\beta(\omega)\exp(-i\omega t)\,d\omega$

$$= \sum_{n=0}^{N-1} \Delta\beta(\omega_n)\exp(-i\omega_n t_m)$$

Numerically, we deal with arbitrary $\Delta\beta$ using a split-step approach to be described shortly

b. When we approximate

$$\beta(\omega_0+\omega) = \beta_0 + \beta_0'\omega + \frac{1}{2}\beta_0''\omega^2 + \frac{1}{6}\beta_0'''\omega^3 + \cdots$$

the convolution simplifies and we obtain:

$$i\frac{\partial u(z,t)}{\partial z} - \frac{1}{2}\beta_0''\frac{\partial^2 u(z,t)}{\partial t^2} - \frac{i}{6}\beta_0'''\frac{\partial^3 u(z,t)}{\partial t^3} + \cdots = 0$$

NOTE: One can only show this in the limit as $T \to \infty$. What are the discretization corrections?

3. The nonlinearity leads to an intensity-dependent phase rotation

Our equation becomes:

$$i\frac{\partial u(z,t)}{\partial z} - \frac{1}{2}\beta_0''\frac{\partial^2 u(z,t)}{\partial t^2} - \frac{i}{6}\beta_0'''\frac{\partial^3 u(z,t)}{\partial t^3} + \gamma|u(z,t)|^2 u(z,t) = 0$$

which we will examine in various limits.

B. Dispersion

1. Setting the nonlinearity and third-order dispersion to zero yields

$$i \frac{\partial u(z,t)}{\partial z} - \frac{1}{2} \beta_0'' \frac{\partial^2 u(z,t)}{\partial t^2} = 0$$

or (equivalently)

$$i \frac{\partial \tilde{u}(\omega, z)}{\partial z} + \frac{1}{2} \beta_0'' \omega^2 \tilde{u}(\omega, z) = 0$$

2. We may now write an explicit solution in the form

$$u(z,t) = \frac{T}{2\pi} \int_0^{\Omega} \tilde{u}(0,\omega) \exp\left[\frac{i}{2}\beta_0'' \omega^2 z - i\omega t\right] d\omega$$

$$\simeq \frac{T}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(0,\omega) \exp\left[\frac{i}{2}\beta_0'' \omega^2 z - i\omega t\right] d\omega$$

when the signal is in a limited bandwidth

a. There a number of cases where $\tilde{u}(0,\omega)$ can be found analytically

Examples: [Extending limits in t to $\pm\infty$]

(1) $u(0,t) = A\exp(-t^2/2\tau^2)$

$$\tilde{u}(0,\omega) = \frac{\sqrt{2\pi}\,A\tau}{T}\exp(-\tau^2\omega^2/2)$$

(2) $u(0,t) = A\,\mathrm{sech}(t/\tau)$

$$\tilde{u}(0,\omega) = \frac{\pi A\tau}{T}\,\mathrm{sech}(\pi\tau\omega/2)$$

These two example are special because the Fourier transform has the same form as the original function

Generally, that is not the case

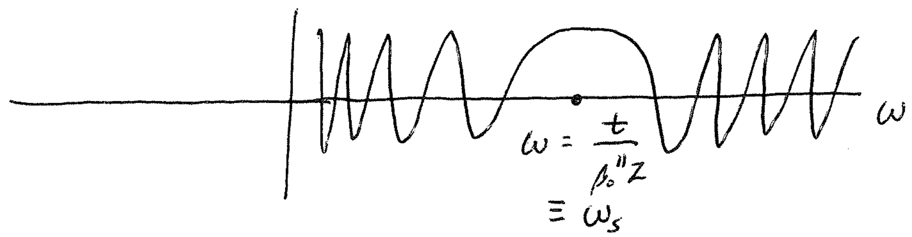
(3) $u(0,t) = \exp(-t/\tau)$ $[t \geq 0]$
$= 0$ $[t < 0]$

$$\tilde{u}(0,\omega) = \frac{A\tau}{T}\,\frac{1}{1-i\tau\omega}$$

b. Only the Gaussian can be solved for analytically for arbitrary z.

More generally, one can use <u>steepest descent</u>

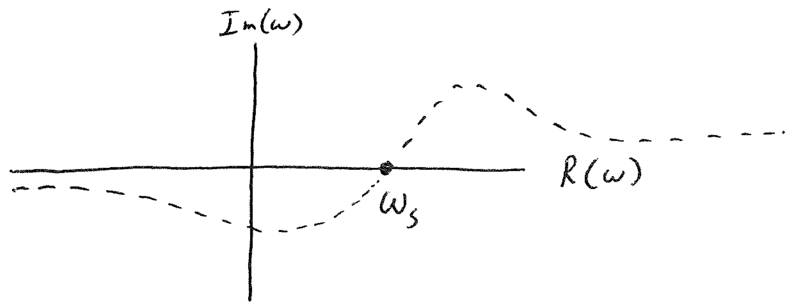
When z is large, the integral oscillates very fast as a function of ω



At each t, only a small region in the neighborhood of the point where the derivative of the argument of the exponent is zero contributes

$$\frac{d}{d\omega}\left[\frac{i}{2}\beta_0''\omega^2 z - i\omega t\right] = i\beta_0''\omega z - it = 0$$

$$\Rightarrow \quad \omega = t/\beta_0'' z \equiv \omega_s$$



Steepest descent path when $\beta_0'' > 0$

We now find

$$u(z,t) \simeq \frac{T}{2\pi}\,\tilde{u}\,(0, t/\beta_0'' z)\exp\left(-it^2/\beta_0'' z\right)$$

$$\cdot \int_{-\infty}^{\infty} \exp\left[\frac{i}{2}\beta_0'' z\,(\omega-\omega_s)^2\right] d\omega$$

(Fresnel integral)

3.1.9

To evaluate, we may let

$(\omega - \omega_s)^2 = \pm i u^2$ (depending on the sign of β_0'')

With this substitution our integral becomes

$$\int_{-\infty}^{\infty} \exp\left[\frac{i}{2}\beta_0'' z (\omega-\omega_s)^2\right] d\omega$$

$$= \sqrt{\frac{2\pi}{|\beta_0''|z}}\ \frac{1 \pm i}{\sqrt{2}}$$

Our final result is

$$u(z,t) \simeq \frac{T}{\sqrt{2\pi|\beta_0''|z}}\ \frac{1\pm i}{\sqrt{2}}\ \tilde{u}(0, t/\beta_0''z)$$

$$\cdot \exp(-it^2/2\beta_0''z)$$

We conclude

(1) All pulses look like their Fourier transforms at large z

(2) There is a $\pi/4$ phase shift at $t=0$.

(3) The phase chirp is given by $-t^2/2\beta_0''z$

(4) The pulse spreads proportionately to $|\beta_0''|z$

(5) Amplitude diminishes proportional to $z^{1/2}$

Note: In the physics convention

This point is one that is easily confusing in the physics convention.

$$\underbrace{\frac{d\varphi}{dt} = -\omega_{local}}_{} = -\omega_s \text{ (in this case)}$$

This is general.

c. Exact result for a Gaussian:

$$u(z,t) = \frac{A\tau}{(\tau^2 - i\beta_0'' z)^{1/2}} \exp\left[-\frac{t^2}{2(\tau^2 - i\beta_0'' z)}\right]$$

$$\sim \frac{A\tau}{\sqrt{|\beta_0''| z}} \frac{1 \pm i}{\sqrt{2}} \exp\left(-it^2/\beta_0'' z\right)$$

at large z, which agrees with general formula.

3. At the zero dispersion point, the third order dispersion dominates and our equation becomes

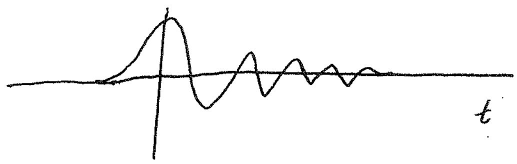
$$\frac{\partial u(z,t)}{\partial z} = \frac{\beta_0'''}{6} \frac{\partial^3 u}{\partial t^3}$$

This equation is purely real

3.1.11

a. Physical interpretation:

At the zero dispersion point, the medium is most transparent and the speed of light is closest to the speed of light in a vacuum. At higher and lower frequencies, the light moves slower, leading to a trailing edge at larger times. The + and − frequencies interfere, leading to oscillations



b. Our evolution integral becomes

$$u(z,t) \simeq \frac{T}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(0,\omega) \exp\left(\frac{i}{6}\beta_0'''\omega^3 z - i\omega t\right) d\omega$$

$$\simeq \frac{T}{2\pi} \tilde{u}(0,0) \cdot \int_{-\infty}^{\infty} \exp\left(\frac{i}{6}\beta_0'''\omega z - i\omega t\right) d\omega$$

Integral is called an Airy integral

$$= \left(\frac{2}{\beta_0''' z}\right)^{1/3} \frac{T}{2\pi} \tilde{u}(0,0) \, Ai\left[-\left(\frac{2}{\beta_0''' z}\right)^{1/3} t\right]$$

Note: β_0''' <u>must</u> be greater than zero!

Note: The shape is dominated ultimately by the Airy function. That is not true at second order, where the shape is dominated by $\tilde{u}$ (but not the phase).

3.1.12

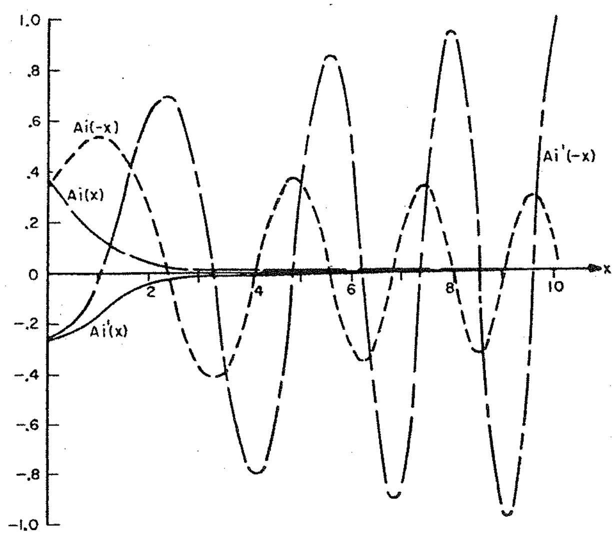


FIGURE 10.6. Ai ($\pm x$), Ai' ($\pm x$).

M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions" (U.S. Department of Commerce Applied Mathematics Series, vol. 55), 1972, p. 446

In general, third-order dispersion leads to a time asymmetry.

C. Nonlinearity

1. In general, nonlinearity leads to complex behavior. There are two key limits where that is not the case

   a. Dispersion is negligible

$$i\frac{\partial u(z,t)}{\partial z} + \gamma\,|u(z,t)|^2 u(z,t) = 0$$

In this case:

$$|u(z,t)|^2 = |u(0,t)|^2$$

from which we conclude

$$u(z,t) = u(0,t)\exp\left[i\gamma|u(0,t)|^2 z\right]$$

The frequency behavior can be found using a steepest descent approach for simple functions.

   b. Solitons

Dispersion is not negligible, but the initial shape is special

3.1.14

$$i\frac{\partial u(z,t)}{\partial z} - \frac{1}{2}\beta_0''\frac{\partial^2 u(z,t)}{\partial t^2} + \gamma|u(z,t)|^2 u(z,t) = 0$$

If $\beta_0'' < 0$ (anomslous dispersion)

then the function

$$u(z,t) = \left(\frac{|\beta_0''|}{\gamma\tau^2}\right)^{1/2}\operatorname{sech}(t/\tau)\exp\left(\frac{i}{2}\frac{|\beta_0''|}{\tau^2}z\right)$$

solves the equation. This pulse does not spread due to dispersion! Hence, the interest for communications

There is no nonlinear distortion

Note that the amplitude and duration are coupled! This is a problem for communications systems.

2. Extension of soliton solution to any frequency

$$u(z,t) = \left(\frac{|\beta_0''|}{\gamma\tau^2}\right)^{1/2}\operatorname{sech}\left[\frac{1}{\tau}(t+|\beta_0''|\omega z)\right]$$
$$\cdot\exp\left(\frac{i}{2}\frac{|\beta_0''|}{\tau^2}z - \frac{i}{2}|\beta_0''|\omega^2 z - i\omega t\right)$$

This amount to the same soliton at a diffeent frequency with a diffeent group velocity

3. Solitons have many other remarkable properties and appear in many contexts

   a. They do not distort when they collide

   b. Deviations from the ideal shape can be treated using perturbation theory.